veloped for its own sake. The thought that some of its results may find application in other sciences is, however, a continual inspiration, and those who investigate such applications sometimes add materially to the development of mathematics.

The curve representing a function of a single variable was the principal object of investigation during the eighteenth and a great part of the nineteenth century.* The investigations of Abel and Cauchy on power series during the early part of the nineteenth century furnished the foundation for the modern theory of analytic functions—a theory which has been adorned by the labors of some of the most brilliant mathematicians of the preceding generation and which is claiming the attention of some of the foremost thinkers of the present time. Quite recently this theory has been made more accessible to English readers by the treatises of Forsyth and Harkness and Morley.

The critical spirit of our age is, in a large measure, due to the study of the theory of analytic functions. "Newton assumes without hesitation the existence, in every case, of a velocity in a moving point, without troubling himself with the inquiry whether there might not be continuous functions having no derivative."† When it was discovered that such functions exist and that the works of some of the greatest mathematicians of the preceding centuries had to be modified in some instances, mathematicians naturally became much more exacting in regard to rigor, and thus ushered in an age which may be compared with the times of Euclid with respect to its demands for rigor. Whether our critical age will produce a work which, like Euclid, will serve as a model for millenniums cannot be foretold, but it seems certain that works which can stand the critical tests of this age will stand the tests of all ages.

The critical spirit of our times is the foundation of what has been styled the *arithmetization of mathematics*. This movement which the late Weierstrass knew so well to lead is pervading more and more the whole mathematical world. We are rapidly banishing from our treatises the term quantity and replacing it by the word number. Our geometric intuitions are forced into the background and logical deductions from intuitions are taking their places. Who can conceive of curves which have no tangent at any of their rational points in a given interval? Nevertheless it is well known that such curves exist. An account of such functions was first published by Hankel in 1870.‡

Mathematicians find themselves in a great dilemma at this point. Geometric intuition has been such a strong instrument of research and has given so much life and beauty to mathematical investigation that mathematicians cling to it as to their own lives. It is an enormous price when rigor can be purchased only with geometric intuition. Yet, in the present stage of mathematical thought, this seems to be the only thing that will be accepted, and mathematicians stand helpless before this decree.

A few examples may throw some light on this subject. What do we un-

*Lie, *Leipziger Berichte*, Vol. 47, page 261.
†Klein, *Evesten Colloquium*, 1894, page 41.
‡Cf. Pierpont, *Bulletin of the American Mathematical Society*, Vol. 5, page 398.

derstand by the length of a continuous curve? The intuitionalist says, if we connect different points of the continuous curve by straight lines and find the sum of the lengths of these straight lines, then the length of the curve will be the limit of this sum as the number of the points is indefinitely increased. Jordan was the first to call attention to the fact that this sum need not have a limit. Hence there are continuous curves which do not have any length according to the ordinary definition of length. In fact a number of area-filling curves have recently been studied, and Cantor has shown that a multiplicity of any number of dimensions can be put in a one to one correspondence with a multiplicity of one dimension.

These are some of the facts which have compelled mathematicians to construct their own worlds—the number worlds. Conclusions drawn in one number world do not necessarily apply to another. When a problem is under consideration the number world is so chosen as to meet the demands of the problem. For instance, the constructions and demonstrations of Euclid's geometry seem to require only a space composed of quadratic numbers.* Hence it appears that we do not need to assume that space is continuous in order to demonstrate the theorems of elementary geometry. Similarly in algebra, we are laying more and more stress upon a distinct statement of the number world (the domain of rationality) in which we are operating. Such specifications add a clearness and rigor to our work which would otherwise scarcely be possible.

This refinement which the mathematical thought of to-day is so actively cultivating is not restricted to the finite region. Mathematical infinity is receiving its share of attention. It is well known that it is sometimes desirable to regard the infinite region as a single point. This is, for instance, the case in the transformation known as inversion. Again, in ordinary projective geometry, it is generally convenient to regard the infinite region as of one lower dimension than the finite, so that the infinite region of a plane is merely a line and the infinite region of space is a plane. The student of differential calculus is, moreover, familiar with the infinite variable and the many simplifications which its use makes possible.

The most fruitful investigations along this line are those on multiplicities (Mengenlehre, ensembles). Any total of definite and clearly defined elements is said to form a multiplicity. If two multiplicities are simply isomorphic, *i. e.*, if there is a 1, 1 correspondence between the elements of the multiplicites, they are said to be equivalent, or to have the same power. For example, it is easy to prove that all the positive rational numbers are equivalent to the natural numbers. To do this we may associate all the rational numbers $p/q$ for which the sum $p+q=n$, any positive integer. We thus have the $n-1$ numbers.

$$\frac{n-1}{1}, \quad \frac{n-2}{2}, \quad \frac{n-3}{3} \cdots, \quad \frac{2}{n-2}, \quad \frac{1}{n-1}.$$

We may let 1 correspond to 1; the numbers for which $n=3$ correspond to

*Cf. Strong, *Bulletin of the American Mathematical Society*, Vol. 4, 1898, page 443.

2 and 3; the numbers for which $n=4$ correspond to 4, 5 and 6, etc. We thus obtain the following 1, 1 corresponding between all the rational numbers and the positive integers:

$$\frac{1}{1};\ \frac{2}{1},\ \frac{1}{2};\ \frac{3}{1},\ \frac{2}{2},\ \frac{1}{3};\ \frac{4}{1},\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{4};\ \frac{5}{1},\ \frac{4}{2},\ \frac{3}{3},\ \frac{2}{4},\ \frac{1}{5};\ \ldots.$$
$$1;\ 2,\ 3;\ 4,\ 5,\ 6;\ 7,\ 8,\ 9,\ 10;\ 11,\ 12,\ 13,\ 14,\ 15;\ \ldots$$

It may be observed that we do not need to reduce the rational fractions to their lowest terms to effect this correspondence. This method of proof is due to Cantor, who has also proved that all algebraic numbers are equivalent to the natural numbers.* How important and far-reaching the investigations along this line are may be inferred from the fact that Jordan has employed them to serve as a foundation of the elementary part of the second edition of his magistral 'Cours d'analyse.'

A large number of mathematical problems may be reduced to equations involving a single unknown. The solution of such equations has occupied a prominent place in the mathematical literature for centuries. The problem is so difficult that it has been attacked from a number of different points and by means of a large variety of instruments. The instrument which has proved to be the most powerful and far-reaching is substitution groups. By means of it Abel succeeded in 1826 to prove that an equation whose degree exceeds four cannot generally be solved by the successive extraction of roots† and Galois a few years later sketched a far-reaching theory of equations which rests upon the theory of these groups.

In recent years it has been recognized (especially through the labors of Sophus Lie) that the theory of groups has very extensive and fundamental application in a large number of the other domains of mathematics. About a year ago the great French mathematician H. Poincaré, showed in an article, published in the Chicago *Monist*,‡ how this concept may be employed in laying the foundations of elementary geometry. It should be observed that the theory of groups is intrinsically based upon the fundamental concepts of mathematics. It is not a superstructure. It stands on its own foundation and supports more or less a number of other mathematical structures.

As this theory is less known than most of the other extensive branches of mathematics it may be desirable to enter into some details. It is evident that there are rational functions of $n$ independent variables $(x_1, x_2, x_3, \ldots\ldots, x_n)$ which are not changed when these variables are permuted in every possible manner. Such functions are said to be symmetric in regard to these variables. The sum of any given power of these variables $(x_1^a+x_2^a+x_3^a+\ldots\ldots x_n^a)$ is an instance of a symmetric function. These functions occupy one extreme. The other extreme is furnished by functions such as $x_1+2x_2+3x_3+\ldots\ldots nx_n$ which change their value for every possible interchange of the variables. Most of the functions are of neither of these extreme types.

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*Cantor, *Crelle*, Vol. 77, page 258; cf. Vol. 84, page 250.
†*Crelle*, Vol. 1.
‡October, 1898.

Suppose that a function $\phi(x_1, x_2, \ldots, x_n)$ is not changed by either of two interchanges of its independent variables. Such interchanges are called substitutions and they may be represented by $S_1$ and $S_2$. Since $\phi$ is not changed by either of the substitutions $S_1$, $S_2$, it cannot be changed by the substitutions which are equivalent to the succession of these substitutions taken in any order. All such substitutions may be represented by $S_1^{\alpha} S_2^{\beta} S_1^{\gamma} S_2^{\delta} \ldots\ldots$* Since only a finite number of permutations are possible with $n$ letters it follows that $S_1^{\alpha} S_2^{\beta} S_1^{\gamma} S_2^{\delta} \ldots\ldots$ can represent only a finite number of distinct substitutions. The totality of these substitutions is said to be a *substitution group*. Hence we observe that every rational function belongs to some substitution group.

It was soon observed that an infinite number of functions belong to the same substitution group and that all of these functions can be expressed rationally in terms of one of them. The researches of Abel, Galois, and Jordan were based upon these facts and they show that the most important problems in the theory of equations involve the theory of substitution groups. The theory of groups was thus founded with a view to its application to a subject of paramount importance. A broad mathematical subject can, however, not grow vigorously and harmoniously as long as it is studied with a view to its direct applications to other mathematical subjects. It must be free to expand in all directions. That freedom for which the human race has ever been struggling must be vouchsafed to such fundamental subjects before they will exhibit their great fertility and far reaching connections. Less than three years ago the first work on the theory of groups that does not consider its application† was given to the public, but the mathematical journals have been publishing a large number of memoirs along the same line for a number of years.

In defining a substitution group we implied only two conditions; viz, no two substitutions of the group are identical and if we combine the substitutions in any way we obtain only substitutions which are already in the group. Substitutions obey *per se* some other conditions; *i. e.*, when they are combined (multiplied together) they obey the associative law and if we multiply a substitution by (or into) two different substitutions the products will be different. In general we say that any finite number of operations which obey these four conditions constitute a group; *e. g.*, all the rotations around the center of a regular solid which make the solid coincide with itself constitute a group, the $n$ $n$th roots of unity constitute a group with respect to multiplication but not with respect to addition, etc.

While the bulk of the mathematicians are reveling in the new fields of thought which are opening up on all sides, without any thought in reference to any immediate practical application of their results, there is fortunately a goodly number whose main efforts are devoted towards making some of these new results useful to the investigators in some of the other sciences. As an instance

*The exponent indicates the number of times the substitution is employed in succession.
†Burnside, *Theory of Groups of a Finite Order*, 1897.

of fairly recent work of the latter kind, we may mention the study of linkages with a view towards describing a straight line. Although the straight line is of such fundamental importance both in pure and applied mathematics, yet it seems it was not until the latter half of the nineteenth century that a mechanical device has been discovered by means of which such a line can be described.

In 1864 M. Peaucellier, an officer of engineers in the French army, discovered the well known device to describe a straight line by means of an instrument composed of seven links. "His discovery was not at first estimated at its true value, fell almost into oblivion, and was rediscovered by a Russian named Lipkin, who got a substantial reward from the Russian government for his supposed originality. However, M. Peaucellier's merit was finally recognized and he has been awarded the great mechanical prize of the Institute of France, the Prix Montyon."*

Although the straight line and the circle occupy such a prominent place in elementary geometry and have been before the eyes of the mathematicians for thousands of years, yet less than half a century has passed since the invention of a mechanical device by means of which the straight line can be drawn. Such discoveries go far towards emphasizing the need of investigations even in the most elementary subjects. Such investigations should, however, be preceded by a thorough knowledge of what has been done along the same lines.

If elementary mathematics is to continue to furnish the best possible preparation for the study of advanced mathematics, it is evident that it has to adapt itself to the rapid changes which are going on in the different branches of mathematics. A need is thus created for elementary text-books which meet the new requirements, and we are happy to be able to state that such books are being produced in our midst. How radical such changes may become cannot be foretold. In his address before the New York Mathematical Society, Simon Newcomb said, "The mathematics of the twenty-first century may be very different from our own; perhaps the schoolboy will begin algebra with the theory of substitution groups, as he might now but for inherited habits."† It is to be hoped that our inherited habits will not furnish an insurmountable barrier to progress in this direction.

In modern times the continent of Europe has always been the most progressive and most of the new theories were first developed in these countries. The theory of invariants seems to be an exception to this rule. The two great English mathematicians, Cayley and Sylvester, developed this theory with great vigor; when their important results became generally known on the continent (largely through the work of Clebsch), they aroused a great deal of interest and they furnished a starting point for many important investigations.

One of the fundamental processes of mathematics is transformation—the deducing of truths from given facts and relations. The expressions which remain invariant when given transformations are performed are naturally objects

*A. B. Kempe, *How to Draw a Straight Line*, page 12.
†*Bulletin of the New York Mathematical Society*, 1894, page 95.

of great interest and of fundamental importance. Imbued with this thought Lie once said, "What do the natural phenomena present to us if it is not a succession of infinitesimal transformations of which the laws of the universe are the invariants?"

It need scarcely be added that all mathematical thought even on the same subject is not running in the same channel. Klein has divided mathematicians into three main categories,* viz, the logicians, the formalists, and the intuitionists. The term logician is "intended to indicate that the main strength of the men belonging to this class lies in their logical and critical powers, in their ability to give strict definitions and to derive rigid deductions therefrom. The great and wholesome influence exerted in Germany by Weierstrass in this direction is well known. The formalists among the mathematicians excel mainly in the skillful formal treatment of a given question, in devising for it an algorithm. Gordan, or let us say Cayley or Sylvester, must be ranged in this group. To the intuitionists, finally, belong those who lay particular stress on geometrical intuition, not in pure geometry only, but in all branches of mathematics. What Benjamin Peirce has called 'geometrizing a mathematical question' seems to express the same idea. Lord Kelvin and von Staudt may be mentioned as types of this category."

In his address before the Zurich International Congress Poincaré says,† "Mathematics has a triple end. It should furnish an instrument for the study of nature. Furthermore, it has a philosophic end, and, I venture to say an esthetic end. It ought to incite the philosopher to search into the notions of number, space, and time; and above all, adepts find in mathematics delights analogous to those that painting and music give. They admire the delicate harmony of numbers and of forms; they are amazed when a new discovery discloses for them an unlooked-for perspective; and the joy they experience has it not the esthetic character although the senses take no part in it? Only the privileged few are called to enjoy it fully, it is true, but is it not the same with all the noblest arts? Hence I do not hesitate to say that mathematics deserves to be cultivated for its own sake and that the theories not admitting of application to physics deserve to be studied as well as others. Moreover, a science produced with a view single to its applications is impossible; truths are fruitful only if they are concatenated; if we cleave to those only of which we expect immediate results the connecting links will be lacking, and there will be no longer a chain."

In closing we may remark that no effort has been made to mention all the new fields of mathematical thought. Mathematics, like the other sciences, seems to offer inexhaustable fields of investigation. As it expands its perimeter increases and hence there is a continually increasing demand for investigators. The fields that have been examined present many difficulties which cannot at present be surmounted. Some of the old difficulties are being removed by the light of the new discoveries. Still we know only a few things even about the

*_The Evanston Colloquium_, page 2.
†_Revue Generale des Sciences_, Vol. 8, page 857.

fields which have been investigated. It is the exception that something can be done by known methods, the rule is that it cannot yet be done.

When we study the literature of some of the older subjects we are sometimes surprised by the large number of known facts, but when we come to study the subjects themselves and ask independent questions we are generally surprised to learn that so few properties are known. Hence it seems very desirable that the advanced student, at least, should study subjects rather than the known facts in regard to these subjects. In this way a more accurate idea of the true state of knowledge can be obtained. Besides the knowledge of having discovered facts and relations which will enter into the structure of a growing science is the greatest source of pleasure that the student can obtain.

*Cornell University, March 9, 1900.*

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## AN ELEMENTARY DERIVATION OF THE SERIES FOR $\sin x$ and $\cos x$.

By H. C. WHITAKER, Ph. D., Philadelphia, Pa.

Assume $\sin x = x + a_3 x^3 + a_5 x^5 + \ldots\ldots\ldots\

Equating coefficients of like powers, $2a_2=-1$, $3a_3=a_2$, $4a_4=-a_3$, and $5a_5=a_4$.

Solving for $a_2$, $a_3$, $a_4$, $a_5$, and substituting in (1) and (2),

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-$$

## INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

[Continued from March Number.]

Writing $s+s^{-1}$ for $\cos 2\varphi$ in $(B)$, $(C)$, $(D)$, $(E)$ we get

$$2(1+e^2-es-es^{-1})^{-\frac{1}{2}}=A_0+A_1(s+s^{-1})+A_2(s^2+s^{-2})+A_3(s^3+s^{-3})+\ldots\ldots+A_n(s^n+s^{-n})+\ldots\ldots\ldots(37).$$

$$2(1+e^2-es-es^{-1})^{-\frac{3}{2}}=B_0+B_1(s+s^{-1})+B_2(s^2+s^{-2})+B_3(s^3+s^{-3})+\ldots\ldots+B_n(s^n+s^{-n})+\ldots\ldots\ldots(38).$$

$$2(1+e^2-es-es^{-1})^{-\frac{5}{2}}=C_0+C_1(s+s^{-1})+C_2(s^2+s^{-2})+C_3(s^3+s^{-3})+\ldots\ldots+C_n(s^n+s^{-n})+\ldots\ldots\ldots(39).$$

$$2(1+e^2-es-es^{-1})^{-\frac{7}{2}}=D_0+D_1(s+s^{-1})+D_2(s^2+s^{-2})+D_3(s^3+s^{-3})+\ldots\ldots+D_n(s^n+s^{-n})+\ldots\ldots\ldots(40).$$

Differentiating (37), (38), and (39) we get

$$e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{3}{2}}=A_1(1-s^{-2})+2A_2(s-s^{-3})+3A_3(s^2-s^{-4})+\ldots\ldots+nA_n(s^{n-1}-s^{-n-1})+\ldots.(41).$$

$$3e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{5}{2}}=B_1(1-s^{-2})+2B_2(s-s^{-3})+3B_3(s^2-s^{-4})+\ldots\ldots+nB_n(s^{n-1}-s^{-n-1})+\ldots.(42).$$

$$5e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{7}{2}}=C_1(1-s^{-2})+2C_2(s-s^{-3})+3C_3(s^2-s^{-4})+\ldots\ldots+nC_n(s^{n-1}-s^{-n-1})+\ldots.(43).$$

From (38) and (41) we get

$$\tfrac{1}{2}e(1-s^{-2})[B_0+B_1(s+s^{-1})+B_2(s_2+s^{-2})+\ldots\ldots+B_n(s^n+s^{-n})+\ldots\ldots]$$
$$=A_1(1-s^{-2})+2A_2(s-s^{-3})+3A_3(s^2-s^{-4})+\ldots\ldots+nA_n(s^{n-1}-s^{-n-1})+\ldots\ldots$$

Equating coefficients of $s^{n-1}$ we get

$$2nAn=e(B_{n-1}-B_{n+1}) \ldots\ldots\ldots\ldots (44).$$

Writing $(n+1)$ for $n$ we get

$$2(n+1)A_{n+1}=e(B_n-B_{n+2}) \ldots\ldots\ldots (45).$$

From (37) and (38) we get

$$(1+e^2-es-es^{-1})[B_0+B_1(s+s^{-1})+B_2(s^2+s^{-2})+\ldots\ldots+B_n(s^n+s^{-n})+\ldots\ldots$$
$$=A_0+A_1(s+s^{-1})+A_2(s^2+s^{-2})+\ldots\ldots+A_n(s^n+s^{-n})+\ldots\ldots$$

Equating coefficients of $s^n$ we get

$$A_n=(1+e^2)B_n-e(B_{n-1}+B_{n+1}) \ldots\ldots\ldots\ldots\ldots (46).$$

Writing $(n+1)$ for $n$ we get

$$A_{n+1}=(1+e^2)B_{n-1}-e(B_n+B_{n+2}) \ldots\ldots\ldots\ldots\ldots (47).$$

Eliminating $B_{n-1}$ between (44) and (46) we get

$$(2n+1)A_n=(1+e^2)B_n-2eB_{n+1} \ldots\ldots\ldots\ldots\ldots (48).$$

Eliminating $B_{n+2}$ between (45) and (47) we get

$$(2n+1)A_{n+1}=-(1+e^2)B_{n+1}+2eB_n \ldots\ldots\ldots\ldots (49).$$

Eliminating $B_{n+1}$ between (48) and (49) we get

$$B_n=\frac{2n+1}{(1-e^2)^2}[(1+e^2)A_n-2eA_{n+1}] \ldots\ldots\ldots\ldots (50),$$

$$B_{n+1}=\frac{2n+3}{(1-e^2)^2}[(1+e^2)A_{n+1}-2eA_{n+2}] \ldots\ldots\ldots (51).$$

From (39) and (42) we get

$$\tfrac{3}{2}e(1-s^{-2})[C_0+C_1(s+s^{-1})+C_2(s^2+s^{-2})+\ldots\ldots+C_n(s^n+s^{-n})+\ldots\ldots$$
$$=B_2(1-s^{-2})+2B_2(s-s^{-3})+\ldots\ldots+nB_n(s^{n-1}-s^{-n-1})+\ldots\ldots$$

Equating coefficients of $s^{n-1}$ we get

$$2nB_n=3e(C_{n-1}-C_{n+1}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (52).$$

Writing $(n+1)$ for $n$ we get

$$2(n+1)B_{n+1}=3e(C_n-C_{n+2}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (53).$$

From (38) and (39) we get

$$(1+e^2-es-es^{-1})[C_0+C_1(s+s^{-1})+C_2(s^2+s^{-2})+\ldots\ldots+C_n(s^n+s^{-n})+ \quad \ldots\ldots$$
$$=B_0+B_1(s+s^{-1})+B_2(s^2+s^{-2})+\ldots\ldots+B_n(s^n+s^{-n})+ \quad \ldots\ldots$$

Equating coefficients of $s^n$ we get

$$B_n=(1+e^2)C_n-e(C_{n-1}+C_{n+1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (54).$$

Writing $(n+1)$ for $n$ we get

$$B_{n+1}=(1+e^2)C_{n+1}-e(C_n+C_{n+2}) \quad \ldots\ldots\ldots\ldots\ldots (55).$$

Eliminating $C_{n-1}$ between (52) and (54) we get

$$(2n+3)B_n=3(1+e^2)C_n-6eC_{n+1} \quad \ldots\ldots\ldots\ldots\ldots (56).$$

Eliminating $C_{n+2}$ between (53) and (55) we get

$$(2n-1)B_{n-1}=-3(1+e^2)C_{n+1}+6eC_n \quad \ldots\ldots\ldots\ldots (57).$$

Eliminating $C_{n+1}$ between (56) and (57) we get

$$C_n=\frac{1}{3(1-e^2)^2}[(2n+3)(1+e^2)B_n-2e(2n-1)B_{n+1}] \quad \ldots\ldots\ldots (58).$$

Similarly,

$$D_n=\frac{1}{5(1-e^2)^2}[(2n+5)(1+e^2)C_n-2e(2n-3)C_{n+1}] \quad \ldots\ldots\ldots (59).$$

And generally,

$$Q_n=\frac{1}{(2m+1)(1-e^2)^2}[(2n+2m+1)(1+e^2)P_n-2e(2n-2m+1)P_{n+1}] \ldots\ldots\ldots (60).$$

(50) and (51) in (58) gives us

$$C_n=\frac{2n+3}{3(1-e^2)^4}[(2n+1)(1+e^2)^2A_n-8en(1+e^2)A_{n+1}+4e^2(2n-1)A_{n+2}] \ldots (61).$$

$$C_{n+1}=\frac{2n+5}{3(1-e^2)^4}[(2n+3)(1+e^2)^2A_{n+1}-8e(n+1)(1+e^2)A_{n+2}$$
$$+4e^2(2n+1)A_{n+3}] \ldots\ldots\ldots\ldots\ldots (62).$$

(61) and (62) in (59) will give us

$$D_n=\frac{2n+5}{15(1-e^2)^6}[(2n+3)(2n+1)(1+e^2)^3A_n-6e(1+e^2)^2(2n+3)(2n-1)A_{n+1}$$
$$+12e^2(1+e^2)(4n^2-5)A_{n+2}-8e^3(2n+1)(2n-3)A_{n+3}] \ldots\ldots\ldots (63).$$

Knowing the values of $A_n$, $A_{n+1}$, etc., we can find the values of $B_n$, $B_{n+1}$, etc.; $C_n$, $C_{n+1}$, etc.; $D_n$, $D_{n+1}$, etc.

[To be Continued.]

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ARITHMETIC.

120. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

How many balls 1 inch in diameter can be put in a cubical box 1 foot in the *clear* each way, putting in the maximum number? [From Greenleaf's *Treatise on Algebra.*]

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa., and H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

The maximum number of balls is not 2149, as given Vol. VII, No. 3, but 2151, as demonstrated below.

Put in 4 rows of 12 balls. Then in the space $8\times12$ can be put 9 more rows of 12 and 11 alternately; for $8\times\frac{1}{11}\sqrt{3}+1=7.928$.

$8-7.928=.072$ of an inch to spare.

This gives in the first layer 9 rows of 12 each $=108$, and 4 rows of 11 each $=44$. $\therefore$ 152 in all.

In the other space $12\times12\times11$ we can put as before eight layers of 144 each and 7 layers of 121 each.

$\therefore$ Eight layers of $144=1152$
Seven layers of $121=847$
One layer of $152=152$

Total $=2151$

125. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

A Quaker once, we understand
For his three sons laid off his land,
And made three equal circles meet
So as to bound an acre neat.
Now in the center of the acre,
Was found the dwelling of the Quaker;
In centers of the circles round,
A dwelling for each son was found.
Now can you tell by skill or art
How many rods they live apart?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The centers of the circles three
With straight lines let united be;
Where touch the arcs, respectively,
These lines will cross the tangency.
Just twice the radius is each line,
And they in trigon space confine
Each circle's sixth and "acre neat,"
No more nor less. With pencil fleet,
From trigon's several vertices
To circles' opposite tangencies,
Respectively, three uprights trace,

And at their intersections place
The Quakers's dwelling. For we find
These uprights are in each combined
Just two-thirds from the trigon-points
And one-third from the tangent-joints.
Each upright we can plainly see
In radius times square root of three; [$r\sqrt{3}$]
And root of three times radius squared [$r^2\sqrt{3}$]
Is trigon's area unimpaired.
A semicircle interjoined
With the Quaker's acre can be coined
An equal to the trigon's space. [$\frac{1}{2}\pi r^2 + 160$]
Now equal to each other place
The areas of the trigon found;
And if the work is true and sound,
We'll find the half of sixty-three [31.50+]
Is a trifle less than in rods should be
The radius of each circled bound
Wherein the sons their dwellings found.
Just twice the radius, or sixty-three, [63.0+]
As the rods apart the sons must be.
Two-thirds of the upright, as shown above,
The sons to their father will have to rove;
This distance, in rods, will two decimals run
In one-eighth of two hundred ninety-one. [36.37+]
Now we've told by skill and rhyming art
The number of rods they live apart.

**II. Solution by J. M. HOWIE, State Normal School, Peru, Neb.; LESLIE J. LOCKE, M. A., Fredonia Institute, Fredonia, Pa.; O. S. WESTCOTT, Chicago, Ill.; J. W. DAPPERT, C. E., Taylorville, Ill.; B. L. REMICK, Bradley Institute, Peoria, Ill.; W. MANZILLA, Langston University, Langston, Okla.**

Let $ABC$ be the triangle formed by joining the centers of the farms, $CE$ the altitude. Since the circles are equal, the triangle $ACB$ is equilateral, and therefore $AC=AB=BC=2r$, where $r$ is the radius of the equal circles.

Area of triangle $ABC=\frac{1}{2}AB\times CE=\frac{1}{2}.2r.\sqrt{(4r^2-r^2)}=r^2\sqrt{3}$.

The area of the three circular sectors included in the triangle$=3.\frac{1}{6}\pi r^2=\frac{1}{2}\pi r^2$.

$\therefore$ The area of the curvilinear triangle $EFI=\sqrt{3}r^2-\frac{1}{2}\pi r^2=160$ sq. rods.

$$\therefore r=\sqrt{\left(\frac{320}{2\sqrt{3}-\pi}\right)}=8\sqrt{\left(\frac{5}{2\sqrt{3}-\pi}\right)}=37.7 \text{ rods.}$$

$2r=AC=75.4$ rods$=$distance between sons' houses, and $\frac{3}{2}\sqrt{3}r=43.5323$ rods$=$distance from father's to sons' house.

Solved in a similar manner by *G. B. M. ZERR, J. SCHEFFER, C. C. CROSS, H. C. WHITAKER, ELMER SCHUYLER, ALOIS F. KOVARIK, JOHN T. FAIRCHILD, J. M. COLAW, HON. JOSIAH H. DRUMMOND, P. S. BERG, H. I. HOPKINS, COOPER D. SCHMITT*, and *J. O. MAHONEY*.

Solutions of problem 124 were received from *CHAS. C. CROSS, P. S. BERG, J. SCHEFFER, G. B. M. ZERR, ELMER SCHUYLER*, and *H. C. WHITAKER*.

## ALGEBRA.

**101. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Prove that $(1+2+3+\ldots+n)+\frac{n}{2!}(1^2+2^2+3^2+\ldots+n^2)+\frac{n(n-1)}{3!}$

$(1^3+2^3+3^3+\ldots+n^3)+\frac{n(n-1)(n-2)}{4!}(1^4+2^4+3^4+\ldots+n^4)+\ldots$

$+\frac{n(n-1)(n-2)}{4!}(1^{n-3}+2^{n-3}+3^{n-3}+\ldots+n^{n-3})+\frac{n(n-1)}{3!}(1^{n-2}+2^{n-2}+3^{n-2})$

$+\ldots+n^{n-2})+\frac{n}{2!}(1^{n-1}+2^{n-1}+3^{n-1}+\ldots+n^{n-1})+(1^n+2^n+3^n+\ldots+n^n)$

$=(n+1)^n-1$, and substitute for $n=2$, 3, 4, 5, and 6.

**Solution by the PROPOSER.**

$$(1+x)^{n+1}=1+(n+1)x+\frac{n(n+1)x^2}{2!}+\frac{n(n+1)(n-1)x^3}{3!}$$

$$+\frac{n(n+1)(n-1)(n-2)x^4}{4!}+\ldots.$$

$$\therefore \frac{(1+x)^{n+1}-1}{n+1}=x\left(1+\frac{nx}{2!}+\frac{n(n-1)x^2}{3!}+\frac{n(n-1)(n-2)x^3}{4!}+\ldots.\right).$$

Let $\frac{n}{2!}$, $\frac{n(n-1)}{3!}$, $\frac{n(n-1)(n-2)}{4!}$, etc.,$=a_1$, $a_2$, $a_3$, etc., respect.

$$\therefore \frac{(1+x)^{n+1}-1}{n+1}=x(1+a_1x+a_2x^2+a_3x^3+\ldots.+a_nx^n).$$

When $x=1$, $\frac{2^{n+1}-1}{n+1}=1+a_1+a_2+a_3+\ldots.+a_n$.

When $x=2$, $\frac{3^{n+1}-1}{n+1}=2+2^2a_1+2^3a_2+2^4a_3+\ldots.+2^{n+1}a_n$.

When $x=3$, $\frac{4^{n+1}-1}{n+1}=3+3^2a_1+3^3a_2+3^4a_3+\ldots.+3^{n+1}a_n$.

. . . . . .

When $x=n$, $\frac{(n+1)^{n+1}-1}{n+1}=n+n^2a_1+n^3a_2+n^4a_3+\ldots.n^{n+1}a_n$

Adding, we get at once,

$$\frac{2^{n+1}+3^{n+1}+4^{n+1}+\ldots.+(n+1)^{n+1}-n}{(n+1)}$$

$=(1+2+3+4+\ldots.+n)+(1^2+2^2+3^2+4^2+\ldots.+n^2)a_1+(1^3+2^3+3^3+4^3$

$+\ldots.+n^3)a_2+\ldots.+(1^n+2^n+3^n+4^n+\ldots.4^n+\ldots.+n^n)a_{n-1}$

$+(1n^{n+1}+2^{n+1}+3^{n+1}+4^{n+1}+\ldots.+n^{n+1})a_n.$

Transposing the last quantity in parenthesis, reducing, and replacing the values of $a_1$, $a_2$, $a_3$, we get,

$$(1+2+3+\ldots.+n)+\frac{n}{2!}(1^2+2^2+3^2+\ldots.+n^2)+\frac{n(n-1)}{3!}(1^3+2^3+3^3+..n^3)$$

$$+\ldots.+\frac{n(n-1)}{3!}(1^{n-2}+2^{n-2}+3^{n-2}+\ldots.+n^{n-2})+\frac{n}{2!}(1^{n-1}+2^{n-1}+3^{n-1}$$

$$+\ldots.+n^{n-1})+(1^n+2^n+3^n+\ldots.+n^n)=(n+1)^n-1.$$

When $n=2$, we get $(1+2)+(1^2+2^2)=3^2-1=8$.

When $n=3$, $(1+2+3)+\frac{3}{2}(1^2+2^2+3^2)+(1^3+2^3+3^3)=4^3-1=63$.

When $n=4$, $(1+2+3+4)+2(1^2+2^2+3^2+4^2)+2(1^3+2^3+3^3+4^3)$
$+(1^4+2^4+3^4+4^4)=5^4-1=624$.

When $n=5$, $(1+2+3+4+5)+\frac{5}{2}(1^2+2^2+3^2+4^2+5^2)+\frac{10}{3}(1^3+2^3+3^3$
$+4^3+5^3)+\frac{5}{2}(1^4+2^4+3^4+4^4+5^4)+(1^5+2^5+3^5+4^5+5^5)=6^5-1=7775$.

When $n=6$, $(1+2+3+4+5+6)+3(1^2+2^2+3^2+4^2+5^2+6^2)+5(1^3+2^3+3^3$
$+4^3+5^3+6^3)+5(1^4+2^4+3^4+4^4+5^4+6^4)+3(1^5+2^5+3^5+4^5+5^5+6^5)+(1^6+2^6+3^6$
$+4^6+5^6+6^6)=7^6-1=117648$.

Also solved by *ELMER SCHUYLER*.

102. Proposed by J. MARCUS BOORMAN, Woodmere, N. Y.

Solve $2x+\sqrt{[x^2-7]}=5$.

I. Solution by COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.; M. A. GRUBER, A. M., Washington, D. C.; J. SCHEFFER, A. M., Hagerstown, Md; and G. B. M. ZERR, A. M., Ph. D., Chester High School, Chester, Pa.

Transposing, $\sqrt{(x^2-7)}=5-2x$.

Squaring, $x^2-7=25-20x+4x^2$; whence $3x^2-20x+32=0$.

Solving, $x=4$ or $\frac{8}{3}$.

Neither of these satisfies the original equation, but by writing it thus, $2x-\sqrt{(x^2-7)}=5$, both values will satisfy it.

II. Solution by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Denote $2x-5$ by $y$; then the given equation reduces to

$$+\sqrt{(y^2+10y-3)}=-2y.$$

Divide by $y$, $$+\sqrt{(1+10/y-3/y^2)}=-2.....(1).$$

But this equation is absurd, since it makes a positive square root equal to a negative number.

*Assume the symbolism* $+\sqrt{\ }j=-1$, *j being an impossible quantity, the root of the equation* $+\sqrt{\ }x+1=0$.

Square (1), $1+10/y-3/y^2=4j$, $y^2(4j-1)-10y+3=0$, from which

$$x=\frac{10j\pm\sqrt{\ }(7-3j)}{4j-1}.$$

III. Solution by the PROPOSER.

Transpose and square.

$\therefore\ 4x^2-20x+25=x^2-7$ .....(*B*), an obvious quadratic.

Apply its roots, 4 and $\frac{8}{3}$, to the given (*A*); hence $2(4)+[-3]=8-3=5$; .....$=2x+\{-[\sqrt{\ }(16-7)]\}$ ..... (*C*); and

$$2(\tfrac{8}{3})+(-\tfrac{1}{3})=5\tfrac{1}{3}-\tfrac{1}{3}=5,\ .....=2x+\{-[\sqrt{\ }(\tfrac{64}{9}-\tfrac{63}{9})]\}.....(D);$$

satisfy it. Could extracting $\sqrt{\ }(x^2-7)$ *positive* here *also* yield roots, then (*A*)'s dominant *quadratic* (*B*) is *bi*-quadratic, which is absurd.

Also solved by *P. S. BERG* and *CHAS. C. CROSS*.

## GEOMETRY.

127. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The equation to the plane through the extremities, $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$, of conjugate diameters of the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is } \frac{x_1+x_2+x_3}{a^2}x+\frac{y_1+y_2+y_3}{b^2}y+\frac{z_1+z_2+z_3}{c^2}z=1.$$

Solution by the PROPOSER.

If $lx+my+nz=p$ ....(1) be the required plane, we should have

$$lx_1+my_1+nz_1=p.....(2),$$
$$lx_2+my_2+nz_2=p.....(3),$$
$$lx_3+my_3+nz_3=p.....(4).$$

Solving these for $l/p$, $m/p$, $n/p$, we have

$$l/p=\begin{vmatrix}1, & y_1, & z_1\\ 1, & y_2, & z_2\\ 1, & y_3, & z_3\end{vmatrix}\div\begin{vmatrix}x_1, & y_1, & z_1\\ x_2, & y_2, & z_2\\ x_3, & y_3, & z_3\end{vmatrix}.....(5).$$

$m/p$=etc., $n/p$=etc., ..... (6).

Reducing (5), making use of

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1=\frac{x_2^2}{a^2}+\ldots\ldots=\frac{x_3^2}{a^2}+\ldots\ldots \quad (7),$$

$$\text{and } x_1y_1+x_2y_2+x_3y_3=y_1z_1+\text{etc.},=z_1x_1+\text{etc.},=0\ldots\ldots \quad (8),$$

$$l/p=\frac{x_1+y_1+z_1}{a^2},\ m/p=\text{etc.},\ n/p=\text{etc.},\ \ldots\ldots \quad (9).$$

These must be put into (1).

Also solved by *G. B. M. ZERR, J. W. YOUNG, LON C. WALKER, J. SCHEFFER,* and *GEORGE LILLEY.*

**128.** Proposed by **W. H. CARTER**, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

Given $F=\triangle^{n-1}\div(n-1)!.\triangle_1.\triangle_2\ldots\ldots\triangle_n$, where $\triangle$=the determinant $(a_1b_2c_3\ldots\ldots k_n)$ and $\triangle_1\triangle_2\ldots\ldots\triangle_n$ are the minors of the elements of the $n$th column; and $a_m, b_m, c_m\ldots\ldots$etc., $(m=1, 2, 3\ldots\ldots n)$ are the coefficients of $n$ given equations containing $n-1$ variables. Show (1) that $n=3$, $F$=the area of a triangle, and (2) if $n=4$, $F$=the volume of the tetrahedron.

Solution by **J. W. YOUNG**, Student in Ohio State University, Columbus, O.

1. Let $n=3$. The points of intersection of the three lines represented by the given equations, are

$$x_1=-\frac{A_1}{C_1};\ x_2=-\frac{A_2}{C_2};\ x_3=-\frac{A_3}{C_3};$$

$$y_1=-\frac{B_1}{C_1};\ y_2=-\frac{B_2}{C_2};\ y_3=-\frac{B_3}{C_3};$$

where, by the usual notation, $A_k$ equals the co-factor $a_k$, in the determinant $\triangle$.

The area of the triangle formed by these points is

$$\text{Area}=\tfrac{1}{2}\begin{vmatrix} x_1, & y_1, & 1\\ x_2, & y_2, & 1\\ x_3, & y_3, & 1\end{vmatrix}=\tfrac{1}{2}\begin{vmatrix} -\frac{A_1}{C_1}, & -\frac{B_1}{C_1}, & 1\\ -\frac{A_2}{C_2}, & -\frac{B_2}{C_2} & 1\\ -\frac{A_3}{C_3}, & -\frac{B_3}{C_3} & 1\end{vmatrix}=\tfrac{1}{2}\begin{vmatrix} A_1, & B_1, & C_1\\ A_2, & B_2, & C_2\\ A_3, & B_3, & C_3\end{vmatrix}\div C_1C_2C_3$$

and this, by a well-known theorem in determinants,

$$=\tfrac{1}{2}\triangle^2\div C_1C_2C_3=F.$$

2. Let $n=4$. The intersections of the four planes given by the equations are found precisely as above.

The volume of the tetrahedron found by the points is

$$\frac{1}{3\,!}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$$

or substituting the values of $x_1 y_1 z_1$, etc., we have

$$\text{Volume} = \tfrac{1}{6}\begin{vmatrix} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \\ A_4, & B_4, & C_4, & D_4 \end{vmatrix} \div D_1 D_2 D_3 D_4$$

$$= \tfrac{1}{6} \triangle^3 \div \triangle_1 \triangle_2 \triangle_3 \triangle_4 = F.$$

Also solved by *G. B. M. ZERR*, *WALTER H. DRANE*, and the *PROPCSEE*.
Professor Carter asks: What does $F$ represent when $n$ is greater than 4?

## CALCULUS.

**97. Proposed by ARTEMAS MARTIN, A.M., Ph.D., LL.D., United States Coast and Geodetic Survey Office, Washington, D. C.**

An auger hole, radius $r$, is bored through a prolate spheroid; the axis of the auger passing through the center, perpendicular to the major axis. Find the volume removed.

**Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Let $\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1$, be the equation to the prolate spheroid.

$x^2 + y^2 = r^2$, the equation to the cylinder.

$$\therefore\ V = 8/a \int_0^r \int_0^{\sqrt{(r^2-x^2)}} \sqrt{[b^2(a^2-x^2) - a^2 y^2]}\, dx\, dy$$

$$= 4/a \int_0^r \sqrt{\{(r^2-x^2)[a^2(b^2-r^2) + (a^2-b^2)x^2]\}}\, dx$$

$$+ \frac{4b^2}{a^2} \int_0^r (a^2-x^2) \sin^{-1}\left[\frac{a}{b}\sqrt{\left(\frac{r^2-x^2}{a^2-x^2}\right)}\right] dx$$

$$= 4/a \int_0^r \sqrt{\{(r^2-x^2)[a^2(b^2-r^2) + (a^2-b^2)x^2]\}}\, dx$$

$$+ \frac{4b^2(a^2-r^2)}{3a} \int_0^r \frac{x^2\, dx}{\sqrt{\{(r^2-x^2)[a^2(b^2-r^2) + (a^2-b^2)x^2]\}}}$$

$$+ \frac{8ab^2(a^2-r^2)}{3} \int_0^r \frac{x^2\, dx}{(a^2-x^2)\sqrt{\{(r^2-x^2)[a^2(b^2-r^2) + (a^2-b^2)x^2]\}}}$$

Let $x=r\cos\theta$, $\frac{r^2(a^2-b^2)}{b^2(a^2-r^2)}=e^2$, $\frac{r^2}{a^2-r^2}=c$.

$$V=\frac{4br^2\sqrt{(a^2-r^2)}}{a}\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta$$

$$+\frac{4br^2\sqrt{(a^2-r^2)}}{3a}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$+\frac{8abr^2}{3\sqrt{(a^2-r^2)}}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore\ V=\frac{8abr^2(c+1)}{3c\sqrt{(a^2-r^2)}}\Pi(e,\ c,\ \tfrac{1}{2}\pi)+\frac{8br^2(a^2-r^2)}{3a}E(e,\ \tfrac{1}{2}\pi)$$

$$+\frac{4br^2}{3ce^2\sqrt{(a^2-r^2)}}[c(1+e+e^2)(a^2-r^2)-2a^2e^2]F(e,\ \tfrac{1}{2}\pi)$$

$$=\frac{8a^3b}{3\sqrt{(a^2-r^2)}}\Pi(e,\ c,\ \tfrac{1}{2}\pi)+\frac{8br^2(a^2-r^2)}{3a}E(e,\ \tfrac{1}{2}\pi)$$

$$+\frac{4b\sqrt{(a-r^2)}}{3ae^2}[r^2(1+e+e^2)-2a^2e^2]F(e,\ \tfrac{1}{2}\pi).$$

Also solved by *J. SCHEFFER*.

## MECHANICS.

**87. Proposed by H. C. WHITAKER. M.E., Ph.D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.**

> "He on his impious foes onward drove,
> Drove them before him to the bounds
> And crystal walls of Heaven; which opening wide
> Rolled inward and a spacious gap disclosed
> Into the wasteful deep; headlong themselves they threw
> Down from the verge of Heaven.
> Nine days they fell; Hell at last
> Yawning received them whole and on them closed."
>
> *Paradise Lost, Book VI.*

Assuming Hell to be the center of the earth, and the only force acting on the lost spirits to be that of gravity due to the earth's attraction,—How far is Heaven ?

**II. Solution by the PROPOSER.**

The published solution of No. 87, Mechanics, assumes (page 82, line 4) that $s=\frac{1}{2}gt^2$ is the formula for falling bodies for great distances.

Neglecting the change in passing from the surface to the center, the formula is $s^3=\frac{gr^2t^2}{2\pi^2}$. (See *Watson's Theoretical Astronomy*, page 46, eq. 28).

This formula gives a result somewhat greater than 90 radii of the earth. Assuming 90 radii as correct, the distance the spirits fell in the first second is,

$$x : 16.1 :: 1^2 : 90^2$$

$x$ being .0237 inch.

Assuming that 460,000 radii was correct, the distance fallen in the first second would be .000,000,000,007,2 inch.

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## NOTE ON PROBLEM 95.

BY FLORIAN CAJORI, PH. D., PROFESSOR OF MATHEMATICS, COLORADO COLLEGE, COLORADO SPRINGS, COLO.

The problem arose in a discussion carried on in the *Nation*, Vol. 68, page 376, between Mr. C. S. Peirce and myself, relating to the validity of an argument given by Galileo and intended to refute the hypothesis that the velocity of a falling body varies as the distance described from a state of rest. Galileo says: "If the velocity with which a body overcomes four yards is double the velocity with which it passed over the first two yards, then the times necessary for these processes must be equal; but four yards can be overcome in the same time as two yards only if there is an instantaneous motion." Mr. Peirce argues that Galileo's reasoning is sound, a claim which I cannot admit.

The assumption that the velocity shall be proportional to the distance described from the state of rest can be expressed by the formula

$$\frac{ds}{dt}=as, \text{ where } a \text{ is a constant.}$$

Hence the acceleration is $\frac{d^2s}{dt^2}=a^2s$. Now initially the distance passed over is zero, *i. e.* $s=0$. Hence the initial accelleration is zero and the body can never begin to move. This conclusion stands even when $a$ is infinitely large, for when absolute zero is a factor, then the product must be zero, no matter how large the other factor may be.* This is the point on which the whole discussion turns. Since Galileo concludes that instantaneous motion is the result, when really there can be no motion at all, his reasoning is fallacious.

But Peirce argues that Galileo used both assumptions stated in the problem, viz., (1) $\frac{ds}{dt}=as$, and (2) $t$ finite for a finite distance. Peirce says: ". . . . the solution of the differential equation $\frac{ds}{dt}=as$ is $s=Ce^{at}$. In order that $s$ and $t$ should both be zero together, $C$ must be infinitesimal. Then, for a finite value of $s$, either $a$ or $t$ must be infinite. That is, either the acquired velocity or the time of fall must be infinite. Galileo's argument first adduces the fact that the time is finite, and on that assumption concludes that the hypothesis would in-

*"In putting together *naughts* to arrive at 1, *we never make any way at all;* the second thousand processes gives no more than the first." DE MORGAN.

volve an infinite acquired velocity, which is absurd." In this way Peirce justifies Galileo's conclusion.

The error in Peirce's reasoning seems to me perfectly apparent. When $s=0$ he takes $C$ to be an infinitesimal, while really $C$ is an absolute zero. When $s$ and $t$ are both zero, $e^{at}$ is not zero, hence $e^{at}$, multiplied by an infinitesimal, cannot be equal to absolute zero. An infinitesimal is a variable whose limit is zero, but the variable never reaches its limit. If we consider an infinitesimal an extremely small quantity, we must still remember that it is a quantity. Now, if $C$ is absolute zero, then $s$ can never be different from zero, no matter how large $e^{at}$ may be.

It is easy to illustrate my conclusion by physical examples. A particle is placed in a smooth tube which revolves horizontally about an axis through its center. With what velocity will the particle move? The only force impelling the particle along the tube is the centrifugal force due to rotation. Hence we have $\frac{d^2s}{dt^2}=w^2s$ and $\frac{ds}{dt}=ws$, where $w$ is the uniform angular velocity. Here the velocity is proportional to the distance from the axis. Suppose now that the particle lies initially at rest *in the axis*. Will it begin to move? There is no reason why it should move one way any more than the other.

The two assumptions in our problem are contraries. The first excludes the possibility of motion; the second declares that motion exists. From assumptions that are contraries no conclusion can be drawn.

## DIOPHANTINE ANALYSIS.

80. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find three square numbers whose reciprocals form an arithmetical progression.

I. Solution by the PROPOSER.

When three numbers are in arithmetical progression, the products of these numbers, taken two at a time, will give three numbers whose *reciprocals* are in arithmetical progression.

Let $x-a$, $x$, and $x+a$=three numbers in arithmetical progression. Then $x(x-a)$, $(x-a)(x+a)$, and $x(x+a)$=the three numbers whose reciprocals are in arithmetical progression.

For, $$\frac{1}{x(x-a)}-\frac{1}{(x-a)(x+a)}=\frac{1}{(x-a)(x+a)}-\frac{1}{x(x+a)}=\frac{a}{x(x-a)(x+a)}.$$

The general values for three *squares* in arithmetical progression are $(p^2-q^2)^2$, $2pq)^2$, $(p^2+q^2)^2$, and $(p^2-q^2+2pq)^2$, where $x=(p^2+q^2)^2$, and $a$ the common difference$=4pq(p^2-q^2)$.

Put $p=2$ and $q=1$; then $1^2$, $5^2$, and $7^2$ are three squares in arithmetical progression. Whence the three squares whose reciprocals form an arithmetical

progression are $5^2=1^2\times5^2$, $7^2=1^2\times7^2$, and $35^2=5^2\times7^2$; and the progression is $\frac{1}{25}$, $\frac{1}{49}$, $\frac{1}{1225}$.

Put $p=3$ and $q=2$; then the required progression is $(\frac{1}{91})^2$, $(\frac{1}{119})^2$, $(\frac{1}{221})^2$.

II. Solution by O. S. WESTCOTT, A. M., Sc. D., Maywood, Ill.

Let $x^2$, $y^2$, $z^2$ represent the numbers. Then $1/z^2-1/y^2=1/y^2-1/x^2$. And $1/z^2+1/x^2=2/y^2$, or $y^2(x^2+z^2)=2x^2z^2$, or $y^2/x^2z^2=2/(x^2+z^2)$.

Since the first member of this equation is a square, the second must be. $2/(x^2+z^2)=4[2(x^2+z^2)]$, and we have to make $2(x^2+z^2)$ a square.

Put $x=7$ and $z=1$; then $2(x^2+z^2)=2(49+1)=10^2$. Hence the numbers are 49, $\frac{49}{25}$, 1; the progression being $\frac{1}{49}$, $\frac{25}{49}$, 1.

Or put $x=\frac{1}{7}$ and $z=1$; then $2(x^2+z^2)=2[(\frac{1}{49})+1]=(\frac{10}{7})^2$, and the numbers are $\frac{1}{49}$, $\frac{1}{25}$, 1; the progression being 49, 25, 1.

III. Solution by SYLVESTER ROBINS, North Branch, N. J.

Let $a^2$, $x^2$ and $b^2$ represent three squares whose reciprocals $1/a^2$, $1/x^2$, and $1/b^2$ are in arithmetical progression.

Then $1/a^2+1/b^2=2/x^2$, and $x^2=2a^2b^2/(a^2+b^2)$, a square.

Expand $1/2=1, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \frac{1393}{985}, \frac{8119}{5741}$, etc.

Say $a=1$; $b=7$, 41, 239, 1393, 8119, etc.

Then $1^2, (2\times1^2\times7^2)/(1^2+7^2), 7^2$ .....$1, \frac{49}{25}, 49$.

$1^2, (2\times1^2\times41^2)/(1^2+41^2), 41^2$.....$1, \frac{1681}{841}, 1681$.

$1^2, (2\times1^2\times239^2)/(1^2+239^2), 239^2$.....$1, \frac{57121}{28561}, 57121$.

*COOPER D. SCHMITT* and *G. B. M. ZERR* refer to Problem 78. See solutions of that problem in MONTHLY for March, pages 82–83, by *CHARLES C. CROSS, JOSIAH H. DRUMMOND, M. A. GRUBER*, and *G. B. M. ZERR*, who also solved the above problem.

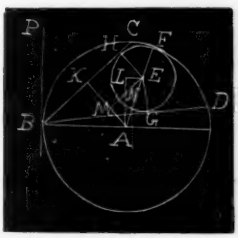
## AVERAGE AND PROBABILITY.

84. Proposed by L. C. WALKER, Associate Professor of Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

From a point in the circumference of a circle two chords are drawn; find (1) the average radius, and (2) the average area of the circle which touches the two chords and the given circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $AB=AF=r$, $HE=GE=FE=x$, $\angle PBD=\theta$, $\angle PBC=\varphi$.




Then $BD=2r\sin\theta$, $BC=2r\sin\varphi$, $AK=r\cos\varphi$, $AN=r\cos\theta$, $BH=BG=BK+KH=BN+NG$.

$KH=ME=\sqrt{[(r-x)^2-(r\cos\varphi-x)^2]}$

$=\sqrt{[r^2\sin^2\varphi-2rx(1-\cos\varphi)]}$.

$NG=LE=\sqrt{[(r-x)^2-(r\cos\theta+x)^2]}$

$=\sqrt{[r^2\sin^2\theta-2rx(1+\cos\theta)]}$.

$\therefore\ r\sin\varphi+\sqrt{[r^2\sin^2\varphi-2rx(1-\cos\varphi)]}$

$$=r\sin\theta+\sqrt{[r^2\sin^2\theta-2rx(1+\cos\theta)]}.$$

$$\therefore\ x=\frac{2r\,[(\sin\theta-\sin\varphi)\sin(\theta+\varphi)-(\sin\theta-\sin\varphi)^2]}{(\cos\theta+\cos\varphi)^2}$$

$$=2r[\sin\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi)\sec^2\tfrac{1}{2}(\theta-\varphi)-\tan^2\tfrac{1}{2}(\theta-\varphi)].$$

$$\pi x^2=4\pi r^2[\sin\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi)\sec^2\tfrac{1}{2}(\theta-\varphi)-\tan^2\tfrac{1}{2}(\theta-\varphi)]^2.$$

Let $L$=average length, $\triangle$=average area.

$$\therefore\ L=\frac{\int_0^\pi\int_0^\theta x\,d\theta\,d\varphi}{\int_0^\pi\int_0^\theta d\theta\,d\varphi}=\frac{2}{\pi^2}\int_0^\pi\int_0^\theta x\,d\theta\,d\varphi$$

$$=\frac{8r}{\pi^2}\int_0^\pi(\theta\cos^2\tfrac{1}{2}\theta-\sin\theta+\sin\theta\log\sec\tfrac{1}{2}\theta)\,d\theta=\frac{2r}{\pi^2}(\pi^2-8)=.3789r.$$

$$\triangle=\frac{\pi\int_0^\pi\int_0^\theta x^2\,d\theta\,d\varphi}{\int_0^\pi\int_0^\theta d\theta\,d\varphi}=\frac{2}{\pi}\int_0^\pi\int_0^\theta x^2\,d\theta\,d\varphi$$

$$=\frac{32r^2}{3\pi}\int_0^\pi(6\theta\cos^4\tfrac{1}{2}\theta-3\theta\cos^2\tfrac{1}{2}\theta-6\sin\tfrac{1}{2}\theta\cos^3\tfrac{1}{2}\theta+2\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$

$$-12\sin\tfrac{1}{2}\theta\cos^3\tfrac{1}{2}\theta\log\cos\tfrac{1}{2}\theta)\,d\theta=\frac{4r^2}{3\pi}(3\pi^2-28)=.2174\pi r^2.$$

**85.** Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Two points are taken at random in a circle and a chord drawn through them; a point is then taken at random in each segment. Find the average area of the quadrilateral formed by joining the four points.

Solution by the PROPOSER.

Let $P$, $Q$, $R$, $S$ be the four random points taken as indicated in the problem, $NN'$ the chord through $PQ$, $MM'$, $TT'$ the chords through $S$, $R$, respectively.

Draw $CD$ perpendicular and $CD'$ parallel to $NN'$. Let $CD=r$, $CE=u$, $CF=v$, $CG=w$, $NQ=x$, $PQ=y$, $NF=\sqrt{(r^2-v^2)}=z$, $TG=\sqrt{(r^2-w^2)}=t$, $ME=\sqrt{(r^2-u^2)}=s$, $\angle D'CA'=\theta$.

An element of the circle at $P$ is $dvdx$; at $Q$, $yd\theta dy$; at $R$, $2tdw$; at $S$, $2sdu$.

The limits of $\theta$ are 0 and $\frac{1}{2}\pi$ and doubled; of $v$, $+r$ and $-r$; of $x$, $2z$ and 0; of $y$, 0 and $x$ and doubled; of $w$, $v$ and $r$ and doubled; of $u$, $-r$ and $v$ and doubled. Area $PRQS=\frac{1}{2}y(w-u)$.

The whole number of ways four points can be taken in the circle is $\pi^4r^8$.

$$\therefore \triangle=\frac{16}{\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}\int_0^{2z}\int_0^{x}\int_v^{r}\int_{-r}^{v}\tfrac{1}{2}y(w-u)d\theta dvdxydy2tdw2sdu$$

$$=\frac{8}{3\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}\int_0^{2z}\int_0^{x}\int_v^{r}[4(a^2-v^2)^{\frac{3}{2}}+6vw\sqrt{(a^2-v^2)}+6a^2w\sin^{-1}(v/a)$$
$$+3\pi a^2w]ty^2d\theta dvdxdydw$$

$$=\frac{16}{3\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}\int_0^{2z}\int_0^{x}(a^2-v^2)^{\frac{3}{2}}y^2d\theta dvdxdy$$

$$=\frac{16}{9\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}\int_0^{2z}(a^2-v^2)^{\frac{3}{2}}x^3d\theta dvdx$$

$$=\frac{64}{9\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}(a^2-v^2)^{\frac{3}{2}}d\theta dv=\frac{35r^2}{18\pi^2}\int_0^{\frac{1}{2}\pi}d\theta=\frac{35r^2}{36\pi}.$$

## MISCELLANEOUS.

**74. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.**

The longest diameter of a horizontal ellipse is $CB=2a=6$ feet. Its shortest diameter is $EF=2b=4$ feet, their intersection being at $D$. Find in an indefinite vertical plane passing through $CB$, a point $A=5$ feet$=c$ from $D$, the ellipse being seen from $A$ as a circle.

**III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

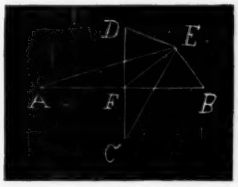
To find a point $E$ in the vertical plane $AEB$ at which $DC$, $AB$ subtend the same angle, we proceed as follows: Let $DF=FC=b=2$, $AF=FB=a=3$, $FE=c=5$, $AE=x$, $BE=y$, $DE=CE=z$, $\angle DEC=\angle AEB=\theta$.

Then $c=\frac{1}{2}\sqrt{(2x^2+2y^2-4a^2)}=\frac{1}{2}\sqrt{(x^2+y^2+2xy\cos\theta)}$.

$\therefore 2(a^2+c^2)=x^2+y^2$ ; $4c^2=x^2+y^2+2xy\cos\theta$.

$\therefore x^2+y^2=2a^2+2c^2$ ..... (1).

$2xy\cos\theta=2c^2-2a^2$ ..... (2).




But $c=\frac{1}{2}\sqrt{(4z^2-4b^2)}=\frac{1}{2}\sqrt{(2z^2+2z^2\cos\theta)}$.

$\therefore z=\sqrt{(b^2+c^2)}$ and $\cos\theta=\frac{c^2-b^2}{c^2+b^2}$.

$$\therefore 2xy=\frac{2(c^2-a^2)(c^2+b^2)}{c^2-b^2}. \quad \therefore x+y=\frac{2\sqrt{(c^4-a^2b^2)}}{\sqrt{(c^2-b^2)}},\ x-y=\frac{2c\sqrt{(a^2-b^2)}}{\sqrt{(c^2-b^2)}}.$$

$$\therefore\ x=\frac{\sqrt{(c^4-a^2b^2)}+c\sqrt{(a^2-b^2)}}{\sqrt{(c^2-b^2)}},\quad y=\frac{\sqrt{(c^4-a^2b^2)}-c\sqrt{(a^2-b^2)}}{\sqrt{(c^2-b^2)}}.$$

Substituting numbers we get $\cos\theta=\frac{2}{29}\sqrt{1}$, $\theta=43^\circ\ 36'\ 9''$.

$$x=\frac{\sqrt{(589)}+5\sqrt{(5)}}{\sqrt{(21)}}=7.73575 \text{ feet},\quad y=\frac{\sqrt{(589)}-5\sqrt{(5)}}{\sqrt{(21)}}=2.85625 \text{ feet}.$$

$z=\sqrt{(29)}=5.38516$ feet.

IV. Solution by A. H. BELL, Hillsboro, Ill.

Let the vertical plane be $ABK$, $A$ the vertex and $BK$ the base of a right cone. The horizontal cutting plane $BC$ is the major axis of the ellipse with $D$ the projection of the minor axis, the cutting plane $GI$ passing through $D$, and parallel to the base $BK$, is a circle and contains the minor axis of the ellipse. Revolving the circle 90° with the diameter $GI$ as an axis, the chord $EF$ is the minor axis of the ellipse; and $s$, $s'$ are the foci. $LC$ is a circle and parallel to the base $BK$ of the cone.

$BD=DC=a=3$ feet, $ED=DF=b=2$ feet, $AD=c=5$ feet.

The properties of an ellipse give $s$, $s'=BL=CK$.....(1).

$$BK\times CL=EF^2=4b^2 \text{ .....} (2).$$

$BC^2=BL^2+BK\times CL$. $\therefore\ BL=2(a^2-b^2)^{\frac{1}{2}}=4.4721360$.....(3).

In the right triangle $ADF$, $AF=AI=AG=(b^2+c^2)^{\frac{1}{2}}=\sqrt{(29)}=5.3851648$.

$BG=GL=CI=\frac{1}{2}BL=2.2360680$.

$AC=AI-CI$. $AB=AI+CI=7.6212328$, $AC=3.1490968$, and the point $A$ is determined.

NOTE. Radius, $GH=(\frac{9}{6})^{\frac{1}{2}}=2.198484326+=\left(\frac{b^2(b^2+c^2)}{2b^2+c^2-a^2}\right)^{\frac{1}{2}}$.

$DH=(GH^2-b^2)^{\frac{1}{2}}=(\frac{5}{6})^{\frac{1}{2}}=0.9128709$.

75. Proposed by J. C. NAGLE, A. M., M. C. E., Professor of Civil Engineering, State Agricultural and Mechanical College, College Station, Texas.

The water tank at the Nacogdoches River on the H. E. & W. T. Ry. is filled by a 3-inch pipe from a reservoir in which the water level is 6 feet above water in tank when full. The top diameter of tank is 17 feet, the bottom diameter is 19 feet, 8 inches, and the pipe projects 10 inches through the bottom. The depth is 13 feet, 6 inches. Find the time required to fill tank, taking the pipe as clean and free from sharp bends, except the right-angled one directly under tank. This bend is 12 feet below outlet of pipe, so that the total length of pipe is 1972 feet. Compare the result with the time of filling if the inlet pipe projected over top of tank instead of entering at the bottom.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Altitude of cone with top of tank as base$=86\frac{1}{16}$ feet; altitude of cone with bottom of tank as base$=99\frac{9}{16}$ feet; altitude of cone with section 10 inches from bottom as base$=98\frac{3}{4}\frac{5}{8}$ feet.

$V$=volume of tank$=\pi\{[(59)^3-(51)^3]/(4)^3\}=1136.375\pi$ cubic feet; $V_1$=volume of portion 10 inches from bottom$=\pi\{[(4779)^3-(4739)^3]/(324)^3\}=79.90614551\pi$ cubic feet. $V_2=V-V_1=1056.46885449$ cubic feet.

Let $v$=velocity of discharge in feet per second, $d$=diameter of pipe in feet, $l$=length, $h$=head of water, $W$=weight of water in pounds discharged per second, $f$=coefficient of friction, $\beta$=coefficient of resistance at entrance of pipe, $\delta$=coefficient of contraction at elbow.

Then $h=(v^2/2g)[(fl/d)+\beta+\delta+1]$. (Hydromechanics).

$\therefore v=\sqrt{\{2ghd/[fl+d(\beta+\delta+1)]\}}$.

$g=32.16$, $l=1972$, $d=\frac{1}{4}$, $f=.030268$, $\beta=.505$, $h=18\frac{3}{4}$, $\delta=.9846$.

$\therefore v=.516351\sqrt{h}$. $Q=\frac{1}{4}\pi d^2 v=.034858\pi$ cubic feet, quantity discharged per second.

$W=62\frac{1}{2}Q=MQ$.

If $n$ is the distance of the top of the tank above the outflow, the resistance or pressure of water in tank on outflow

$$=\tfrac{1}{2}M(6\tfrac{1}{4}\pi)\int_0^n dx=\frac{\pi Mn}{128}.$$

$$\therefore hW=\frac{v_1^2}{2g}\left[\left(\frac{fl}{d}+\beta+\delta+1\right)W+\frac{\pi Mn}{128}\right].$$

$$\therefore 2gh=v_1^2\left(\frac{fl}{d}+\beta+\delta+1+\frac{\pi n}{128Q}\right).$$

$\therefore v_1=2.21788$, since $n=12\frac{3}{4}$. $\therefore Q_1=\frac{1}{4}\pi d^2 v_1=.034654\pi$ cubic feet per second.

$T$=time$=(V_1/Q)+(V_2/Q_1)=(79.90614551\pi/.034858\pi)+(1056.46885449\pi/.034654\pi)$.

$\therefore T=32778.53$ seconds$=9$ hours, 6 minutes, 18.53 seconds.

If filled from the top we will not consider the bend, but suppose pipe 1960 feet long and $h=6$ feet.

Then $h=(v_2^2/2g)[(fl/d)+\beta+1]$.

$\therefore v=\sqrt{\{2ghd/[fl+d(\beta+1)]\}}=1.2713$.

$\therefore Q_2=\frac{1}{4}\pi d^2 v_2=.01986\pi$ cubic feet.

$t$=time$=(V/Q_2)=1136\ 375\pi/.01986\pi=57219.3$ seconds.

$\therefore t=15$ hours, 53 minutes, 39.3 seconds. $t-T=6$ hours, 47 minutes, 20.77 seconds.

II. Solution by P. H. PHILBRICK, M.S., C.E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Lake Charles, La.

First. To find the time of filling the lower 10 inches of the tank. The head is $h=13.5+6-\frac{5}{6}=18\frac{2}{3}$ feet. Let $v$=velocity at the outlet, and $s$=the area of the cross-section of the pipe$=(\frac{1}{4}\pi)(\frac{1}{4})^2=.0491$.

Then $h=v^2/2g+m(v^2/2g)$, or $v=[2gh/(1+m)]^{\frac{1}{2}}$ .....(1), in which $m$ is the sum of the resistances to the entrance and movement of water in the pipe.

The diameter of the tank 10 inches from the bottom is 19.5 feet. Hence the volume of the lower 10 inches of the tank$=(1\frac{1}{2}\pi)[(19.67)^2+(19.5)^2+19.67\times19.5]=V$.....(2).

Then from (1) and (2), $t=V/vs$..... (3).

According to *Bowser*, Articles 110 and 112, $m$ may be taken$=.03\times1972\div\frac{1}{4}+1.5+.98=238.48$.

Second. To find the time of filling the remainder of the tank. Let $x$ be the height in feet of the top of the reservoir above the surface of the water at the time $t$. The diameter of the tank at the surface of the water is $17+(\frac{1}{8}\frac{6}{1})(x-6)=15.815+.1975x$. Hence the area is $A=(\frac{1}{4}\pi)(15.815+.1975x)^2=196.4+4.906x+.0306x^2$. Let $dx$ be the rise in the water in the time $dt$. Then $Adx$ is volume of water admited in time $dt$ ....(4).

We also have, as shown above, $v=[2gx/(1+m)]^{\frac{1}{2}}$, and, therefore, the volume of water admitted in time $dt$ is equal to $[2gh/(1+m)]^{\frac{1}{2}}\times sdt$.....(5).

From (4) and (5) we have, $dt=[(1+m)/2g]^{\frac{1}{2}}\times1/s=[(196.4+4.906x+.0306x^2)dx]/x^{\frac{1}{2}}$ .....(6).

Integrating between $x=4=18\frac{2}{3}$ and $x=h=6$, we have $t'=[(1+m)/2g]^{\frac{1}{2}}\times.0491[392.8(H^{\frac{1}{2}}-h^{\frac{1}{2}})+3.271(H^{\frac{3}{2}}-h^{\frac{3}{2}})+.0122(H^{\frac{5}{2}}-h^{\frac{5}{2}})$.....(7).

$t+t'$=the total time required to fill the tank.

Third. If the inlet pipe projected over the top of the tank, we would have, as in equation (1), $v_1=[2gh_1/(1+m)]^{\frac{1}{2}}$ .....(8), in which $h_1=6$ feet.

Also volume of tank is $V=(\frac{1}{4}\pi)[(19.67)^2+17^2+17\times19.67]$.....(9).

Then $T=V/v_1s$.....(10).

76. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that

$$\log[x-a-b\sqrt{(-1)}]=\tfrac{1}{2}\log[(x-a)^2+b^2]-\sqrt{(-1)}\tan^{-1}\frac{b}{x-a},$$

Naperian logarithms being used.

I. Solution by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Tex.; F. ANDEREGG, A. M., Oberlin College, Oberlin, O.; ARTHUR C. LUNN, University of Chicago, Chicago, Ill; COOPER D. SCHMITT, A.M., University of Tennessee, Knoxville, Tenn.; BURKE SMITH, University of Washington, Seattle, Wash.; H. C. WHITAKER, A. M., Ph. D., Manual Training School, Philadelphia, Pa.

Let $(x-a)-ib=r(\cos\phi-i\sin\phi)$.

Then $r^2=(x-a)^2+b^2$, $\tan\phi=b/(x-a)$.

Thus $\log[x-a-ib]=\log[r(\cos\phi-i\sin\phi)=\log r+\log e^{-i\phi}=\log r-i\phi$

$=\frac{1}{2}\log[(x-a)^2+b^2-i\tan^{-1}[b/(x-a)]$.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Ia.; WALTER H. DRANE, Harvard University, Cambridge, Mass.; and G. B. M. ZERR, A. M., Ph. D., Chester High School, Chester, Pa.

$$\int \frac{\sqrt{(-1)}b\,dx}{(x-a)^2+b^2} = -\sqrt{(-1)}\tan^{-1}\frac{b}{x-a} = \tfrac{1}{2}\log\left(\frac{x-a-b\sqrt{(-1)}}{x-a+b\sqrt{(-1)}}\right).$$

$$\therefore \tfrac{1}{2}\log\left(\frac{x-a-b\sqrt{(-1)}}{x-a+b\sqrt{(-1)}}\right) = -\sqrt{(-1)}\tan^{-1}\frac{b}{x-a}.$$

$$\therefore \log[x-a-b\sqrt{(-1)}] - \tfrac{1}{2}\log[(x-a^2)+b^2] = -\sqrt{(-1)}\tan^{-1}[b/(x-a)].$$

$$\therefore \log[x-a-b\sqrt{(-1)}] = \tfrac{1}{2}\log[(x-a)^2+b^2] - \sqrt{(-1)}\tan^{-1}[b/(x-a)].$$

## PROBLEMS FOR SOLUTION.

### ARITHMETIC.

130. Proposed by H. C. WHITAKER, M.E., Ph.D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

How many balls 1 inch in diameter can be put in a cubical box 2 feet in the clear each way, putting in the maximum number?

131. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

A right frustum of a cone whose radii of the bases are $r$ and $s$, $r>s$, is to be divided into $n$ parts of equal volume by sections parallel to the bases. What are the altitudes of the respective parts?

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

### GEOMETRY.

144. Proposed by L. C. WALKER, Assistant in Mathematics in Leland Stanford, Jr., University, Palo Alto, Cal.

Find the equations of four cones that pass through three given straight lines intersecting in the same point.

145. Proposed by FRANK GRIFFIN, Graduate Student, State University, Boulder, Colo.

If $A$ and $B$ be the points of contact, upon two circles $X$ and $Y$, of tangents drawn from any point of their circle of similitude, then the tangent from $A$ to $Y$ is equal to the tangent from $B$ to $X$. [From *Casey's Sequel to Euclid*, Part I., page 114.]

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

### AVERAGE AND PROBABILITY.

95. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Three random points are taken in an ellipse, one on each side of the major axis and the third anywhere in the ellipse. Find the average area of the triangle formed by joining the three points.

96. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

A random straight line is drawn across a square; find the chance that it intersects two opposite sides. [From *Byerly's Integral Calculus*, page 209].

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

## MECHANICS.

107. Proposed by M. E. GRABER, Student, Heidelberg University, Tiffin, O.

Two particles attracting each other inversely as the square of their distance apart, are constrained to move in straight lines which intersect each other at right angles. How long will it take for the particles to meet and how far does each particle *move ?*

108. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Can it be shown, as a result of Kepler's third law, that the distances are inversely proportional to the squares of the velocities ?

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

## *BOOKS AND PERIODICALS.*

*The Teaching of Elementary Mathematics.* By David Eugene Smith, Principal of the State Normal School at Brockport, New York. 8vo. Cloth, xv+312 pages. Price, $1.00. New York: The Macmillan Company.

In this book may be found the answer to such questions as these: Whence came the subject of mathematics? Why am I teaching it? How has it been taught? What should I read to prepare for my work? Any book which answers these questions to the entire satisfaction of the teacher and the student is worthy of a high place in the category of educational works. One will find in this work an excellent treatment of the various methods of teaching arithmetic, algebra, and geometry, and in this respect it is of inestimable value to all teachers of elementary mathematics.

Not only is this book of great value to teachers of mathematics, but to all those who have under their direction the formulation of courses of study. It is to be hoped that this work will help to dispel that insane notion yet quite prevalent even among professed educators that mathematics has only utilitarian value, that only so much of it ought to be studied as will be used in the general affairs of life. Yet these very same educators see no incongruity in spending five or six years in the study of Greek, or Sanskrit, or Hebrew, or Archeology, even though the student expected to study medicine. It is admitted that a large part of practical arithmetic is not generally applicable to ordinary business, and hence is quite impractical, yet it by no means follows that it may not serve a valuable purpose. "Hamlet may bring us neither food nor clothing, and yet a knowledge of Shakespeare is valuable to every one. It is a matter of no moment in the business affairs of most men that they know where the Caucasus Mountains are, or which way the Rhine flows, or who Cromwell was, and yet we cannot afford to be ignorant of these facts." This is the proper view. Mathematics should be studied and cultivated for its own sake; for the culture and discipline it gives the mind; for the ethical effect its study produces on the mind of man. Since mathematics is becoming more and more the means by which the

phenomena of nature are interpreted, the time is not far distant when one who is ignorant of the principles and processes of the differential and integral calculus will be ignorant of all the advances yet to be made in the fields of natural sciences. Lord Kelvin says that the phenomena of nature are the asymptotes towards which mathematics may approach as near as we please. B. F. F.

*Plane and Solid Geometry—Revised Edition.* By G. A. Wentworth, Author of a Series of Text-Books in Mathematics. 12mo. Half morocco, 386 pages. Price, $1.25. Boston: Ginn & Company. 1899.

This book has stood the supreme test—successful classroom use, and is recognized as one of the very best works published. In the revised edition the author has retained all the strong points and has added some new ones. The printed page is exceedingly attractive. Full, long-dashed, and short-dashed lines of the figures indicate given, resulting, and auxiliary lines, respectively. Bold-faced, italic, and roman type has been skilfully used to distinguish the hypothesis, the conclusion to be proved, and the proof. The reason for each step is now indicated in small type between that step and the one following, thus avoiding the necessity of interrupting the logical train of thought by turning to a previous section. This help is gradually disgarded as the pupil gains more skill and becomes acquainted with a greater number of geometrical truths. The demonstrations are very clear, and the exercises abundant and interesting. J. M. C.

*Algebra for Schools.* By George W. Evans, instructor in Mathematics in the English High School, Boston. 427 pages. Price, $1.12. New York: Henry Holt & Company. 1899.

We note in this book with favor the careful classification of problems, and the emphasis given to the several types of equations arising from them; the scheduled explanation of steps in the reduction of equations; the thorough study of literal equations and generalized problems; the application of factoring to the solution of equations; the separation of Elimination into two parts suitable for linear systems and for linear-quadratic pairs respectively; and the great number of graded examples and problems—some 3500 in all—many of which are new. There is some lack of accuracy of statement. "Algebra is a method of abbreviating the explanations of problems in arithmetic. It is also used to abbreviate the statement of rules and the demonstration of theorems." "If equal qualities are multiplied or divided by the same *amount*, they remain equal." He speaks of "nests" of parentheses. We regard the treatment of negative quantities as especially defective and unscientific. In other respects the work is unusually good in arrangement, material, and method of presentation. J. M. C.

*Infallible Logic.* A visible and automatic system of reasoning. By Thomas D. Hawley, of the Dominion Company, Chicago. 1897.

To the readers of THE AMERICAN MATHEMATICAL MONTHLY the advances made in recent times in Logic are of interest, for they have brought it within the sphere of exact, if not of mathematical analysis. De Morgan started the movement with his *Formal Logic* and his *Memoirs on the Syllogism* and the *Logic of Relatives*. Boole advanced it greatly with his *Laws of Thought*, which he considered his best work; Jevons made a negative advance with his logical machine, and the same may be said of Venn with his *Symbolic Logic*. At the present time Prof. Peano and his colleagues in Italy are applying the Mathematical Logic to express in a concise and exact form all the fundamental definitions and theorems of mathematics. If a young mathematician wishes to become an investigator, he ought by all means to make himself acquainted with the works of the mathematical logicians.

As regards the above work "Infallable Logic," a young mathematician will not get any benefit out of it. It is put forth as a new system, and the author says that "probably

the time is not far distant when it will take the place of the syllogistic and algebraic systems now in current use." It is both amusing and amazing to find that an author who puts forward such claims is ignorant alike of the old logic and of the most elementary mathematics. To prove the former point it is sufficient to mention that he says that the old logic derives by immediate inference from All $S$ is $P$ that Some $P$ is not $S$; and that he does not see the truth of the fundamental principle of all syllogizing, namely that If all $A$ is $B$ and all $B$ is $C$ then all $A$ is $C$. To prove the latter point it is sufficient to mention that he gives as an instance of a real proposition "the angles of any triangle are together equal to three angles," and in making an extract from Jevons he confounds differentiation with deduction and integration with induction.

His system is nothing but Jevons' method of writing down all the possible combinations of a number of qualities and their opposites, striking out the combinations which are negatived by the premises and taking what remains as the conclusion. Jevons wrote the combinations in columns on a "logical slate;" Hawley writes them in the form of a multiplication table, which he calls a Reasoning Frame.

Let us see what this invention of a Reasoning Frame can acomplish. Applied (by the author) to the propositions Washington is the capital of the United States.

Salt is chloride of sodium; it yields the following along with two other equally important conclusions:

Either Washington and the capital of the United States or neither is salt and chloride of sodium or neither.

He applies it to elucidate the fallacy "what you bought yesterday you eat today; you bought raw meat yesterday; therefore you eat raw meat today." He puts $A$=you, $B$=bought, $C$=what. $D$=yesterday $E$=eat, $F$=today, $G$=raw meat, and makes an $ABCD$ $EFG$ table. The conclusion yielded by the Reasoning Frame is rather weak, if indeed, it means anything:

You eat or do not eat today or not today, raw meat. He fails to observe that the combinations he here makes are in no sense whatever the names or characters of one thing.

The Reasoning Frame strikes a snag in the numerically definite syllogism. He quotes the example "If 70 per cent of $M$ are $P$ and 60 per cent are $S$, then at least 30 per cent are both $S$ and $P$; and adds "Of course this reasoning is correct, but I am unable to demonstrate its validity by the Reasoning Frame." As a matter of fact the simplest kind of logical diagram suffices. Let the square represent $M$. Take the most unfavorable case. Suppose that the $M$ which are $p$ are all to the left, and the $M$ which are $s$ are all to the right; there is then 30 per cent of overlapping, hence the $M$ which is both $p$ and $s$ is at least 30 per cent. This principle is written algebraically.

$$sp > s\tfrac{1}{1}p - 100$$

Also $sp < s$ and $< p$.

This single principle is worth more than all that is contained in the "Infallible Logic."

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## NON-EUCLIDEAN GEOMETRY.

By GEORGE BRUCE HALSTED.

In writing of "The Wonderful Century," Alfred Russel Wallace says of all time before the seventeenth century: "Then, going backward, we can find nothing of the first rank except Euclid's wonderful system of geometry, perhaps the most remarkable mental product of the earliest civilizations."

But of late all men of science and intelligent teachers have been hearing more and more of non-Euclidean geometry, and are naturally anxious to know how these new doctrines are related to the traditional geometry which they were taught and perhaps now are teaching.

The new departure is absolutely epoch-making, but fortunately it has intensified admiration for that imperishable model, already in dim antiquity a classic, the immortal *Elements* of Euclid.

But without assumptions nothing can be proved, and Euclid stated his assumptions with the most painstaking candor. He would have smiled at the suggestion that he could ever claim for his conclusions any other truth than perfect deduction from assumed hypotheses.

And so his system is forever safe. Each one of his axioms may turn out to be inconsistent with external reality; each of his fundamental assumptions may be replaced in our final explanation of the space in which we live and move; in reference to our space, all his theorems may be shown to be only approximations; and yet his work will remain a perfect piece of pure mathematics, the exact, eternal geometry of Euclidean space.

For two thousand years no one ever doubted the truth of any one of this

set of axioms, far the most influential in the intellectual history of the world, put together by Euclid in Egypt, but really owing nothing to the Egyptian race, nothing to the boasted lore of Egypt's priests.

The Papyrus of the Rhind, belonging to the British Museum, but given to the world by the erudition of a German Egyptologist, Eisenlohr, and a German historian of mathematics, M. Cantor, gives us more knowledge of the state of mathematics in ancient Egypt than all else previously accessible to the modern world. Its whole testimony confirms with overwhelming force the position that geometry as a science, strict and self-conscious deductive reasoning, was created by the subtle intellect of the same race whose bloom in art still overawes us in the Venus of Milo, the Apollo Belvidere, the Laocoön. But though for twenty centuries the truth of the axioms of the Greek geometer remained unquestioned, there was one of them of which the axiomatic character was doubted even from far antiquity. Elementary geometry was for two thousand years as stationary, as fixed, as peculiarly Greek as the Parthenon. But among Euclid's assumptions is one differing from the others in prolixity, whose place fluctuates in the manuscripts.

Peyrard, on the authority of the Vatican MS., puts it among the postulates, and it is often called the parallel postulate. Heiberg, whose edition of the Greek text is the latest and the best (Leipzig, 1883-1888), gives it as the fifth postulate.

James Williamson, who published the closest translation of the Euclid we have in English, indicating, by the use of italics, the words not in the original, gives this assumption as eleventh among the Common Notions.

Bolyai speaks of it as Euclid's Axiom XI. Todhunter has it as twelfth of the axioms. Clavius (1574) gives it as axiom 13. The Harper Euclid separates it by forty-eight pages from the other axioms.

It is not used in the first twenty-eight propositions of Euclid. Moreover, when at length used, it appears as the inverse of a proposition already demonstrated, the seventeenth, and is only needed to prove the inverse of another proposition already demonstrated, the twenty-seventh.

Geminos of Rhodes (about 70 B. C.) speaks of it as needing proof. The astronomer Ptolemy (A. D. 87-165) tried his hand at proving it. The great Lambert expressly says that Proklus demanded a proof of the assumption because when inverted it is demonstrable. The Arab Nasir-Eddin (1201-1274) tried to demostrate it.

No one had a doubt of the necessary external reality and exact applicability of the assumption. Until the present century the Euclidean geometry was supposed to be the only possible form of space-science; that is, the space analyzed in Euclid's axioms was supposed to be the only non-contradictory sort of space. But could not this assumption be deduced from the other assumptions and the tweth-eight propositions already proved by Euclid without it? Euclid demonstrated things more axiomatic by far. He proves what every dog knows, that any two sides of a triangle are together greater than the third.